Feature Matching via Sparse Relaxation Models

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Content

Introduction

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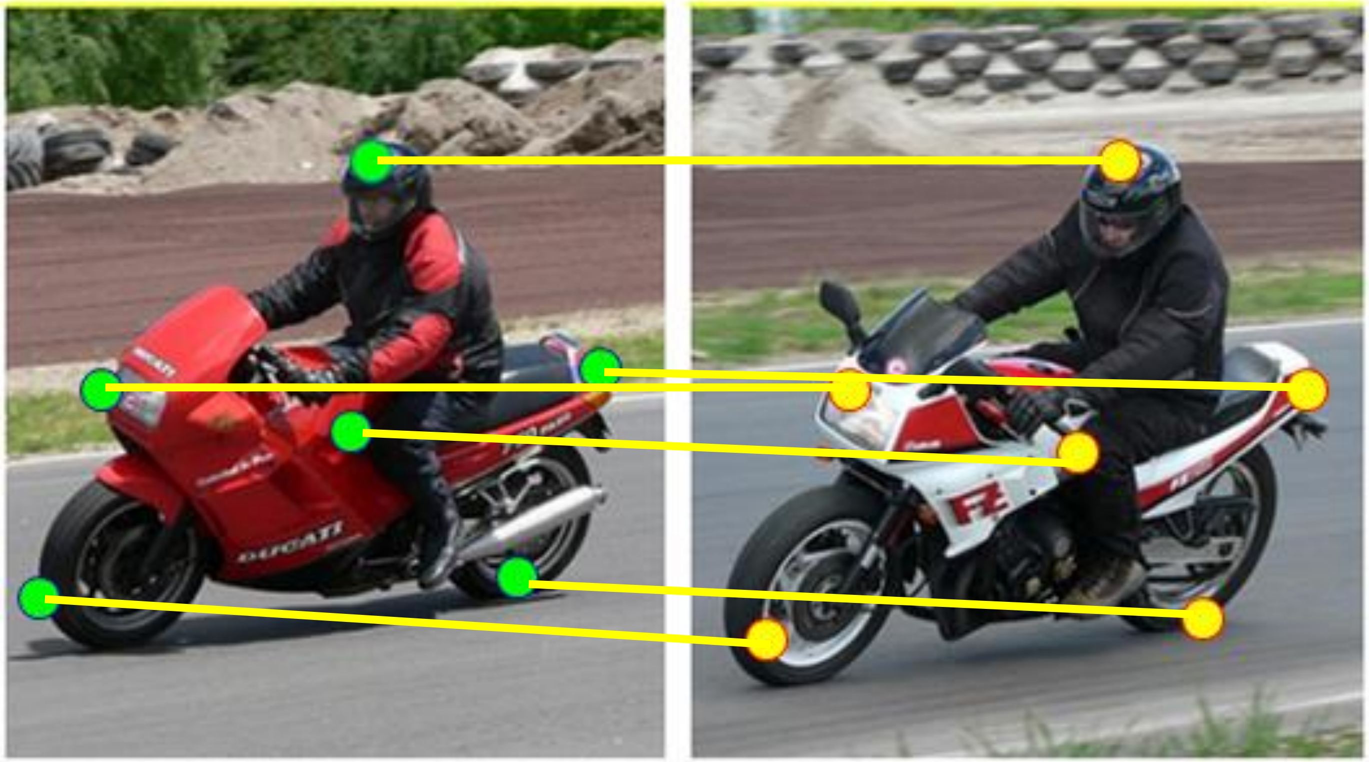

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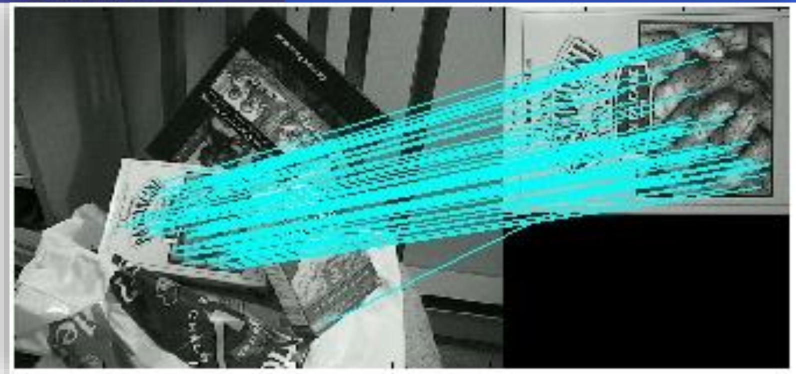
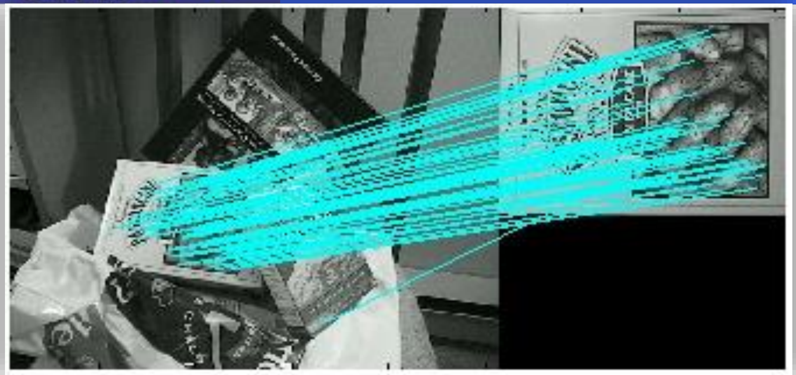


Object detection

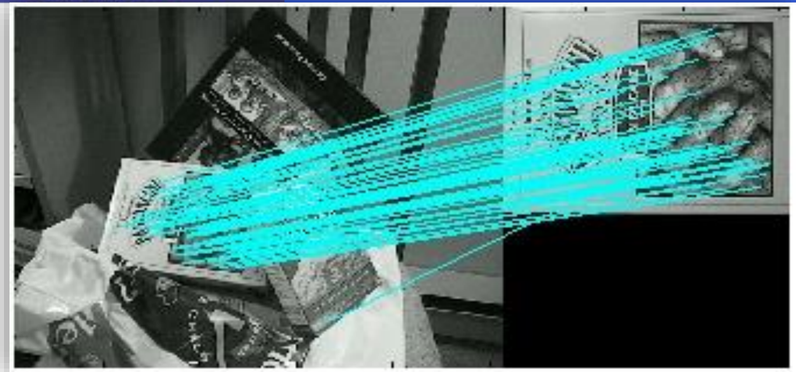
Introduction



Object detection

Person ReID, Zhou et al. AAAI 2018

Introduction



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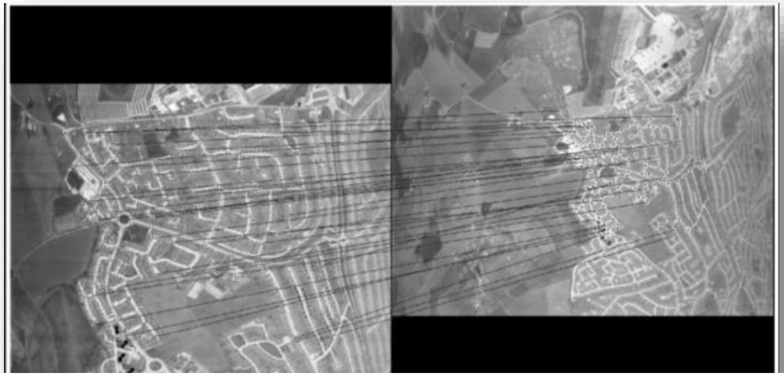
Person ReID, Zhou et al. AAAI 2018



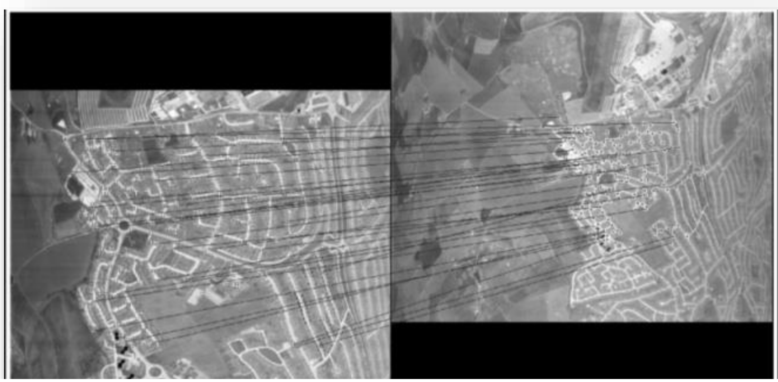
Luo et al. PAMI 2001

Introduction

Object detection

Object tracking, Nebehay et al. CVPR15

Person ReID, Zhou et al. AAAI 2018



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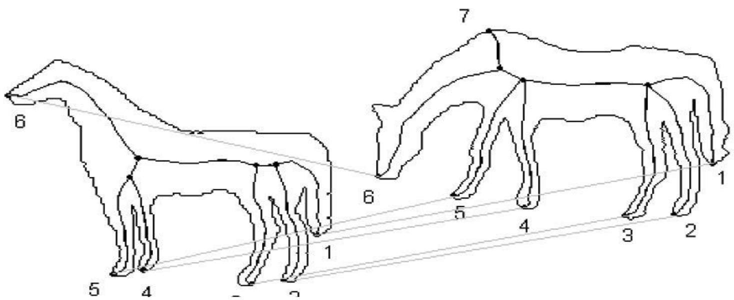
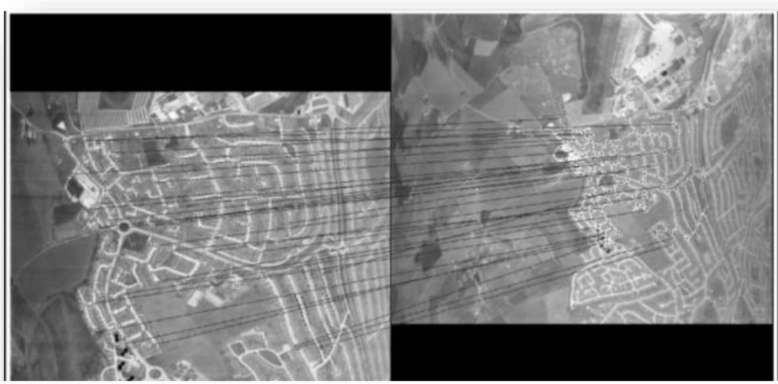
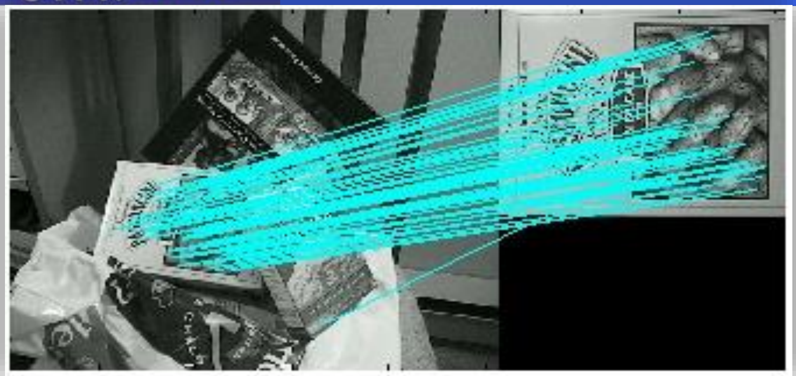


Shape matching, Bai et al. PAMI2008



Luo et al. PAMI 2001

Introduction



Object detection

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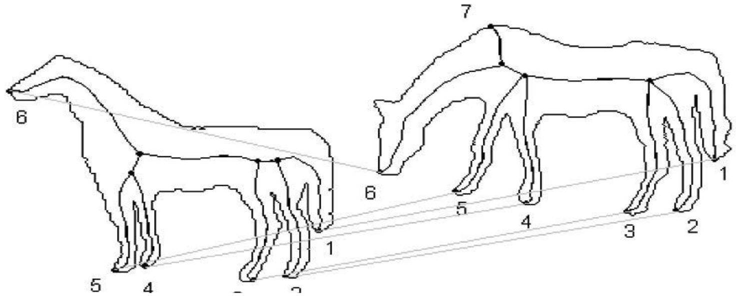
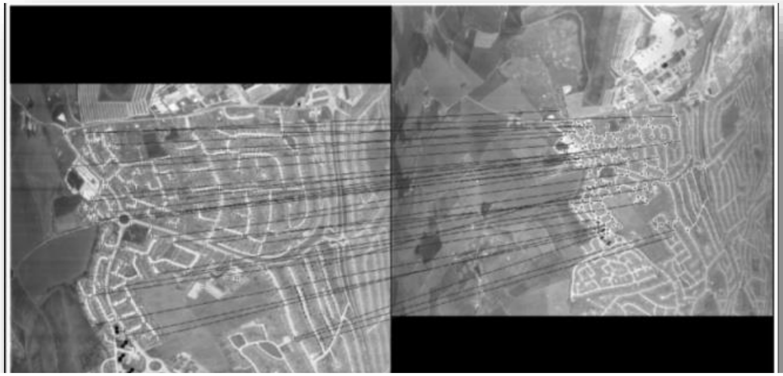
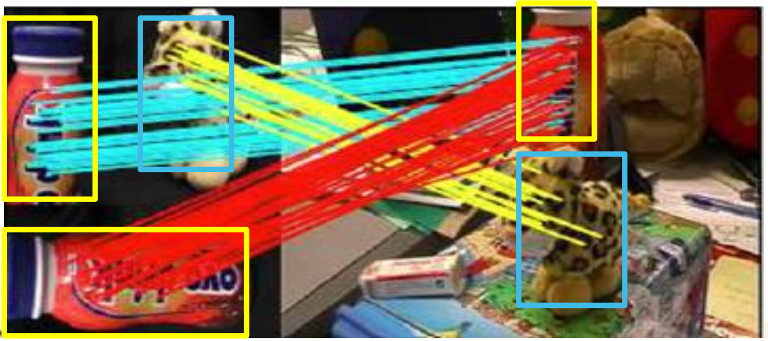


Shape matching, Bai et al. PAMI2008

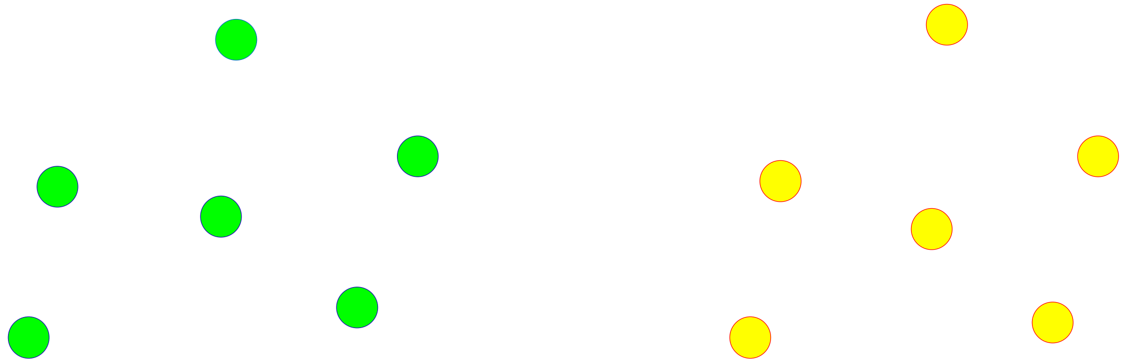


Luo et al. PAMI 2001

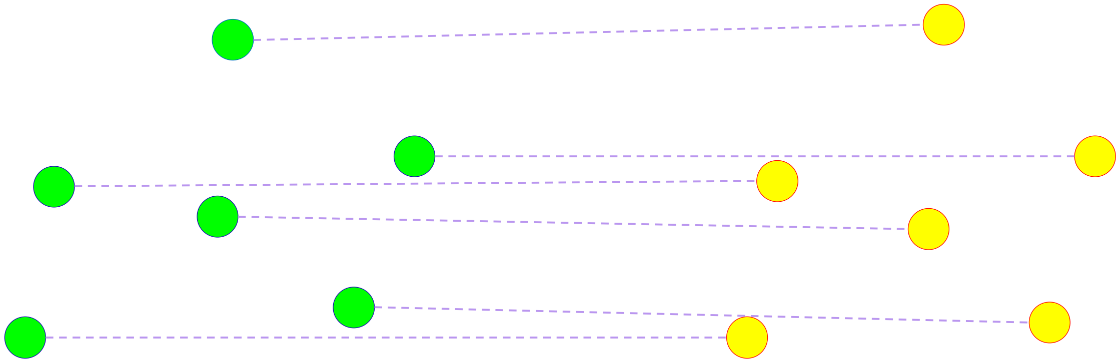


Common Visual Pattern Discovery

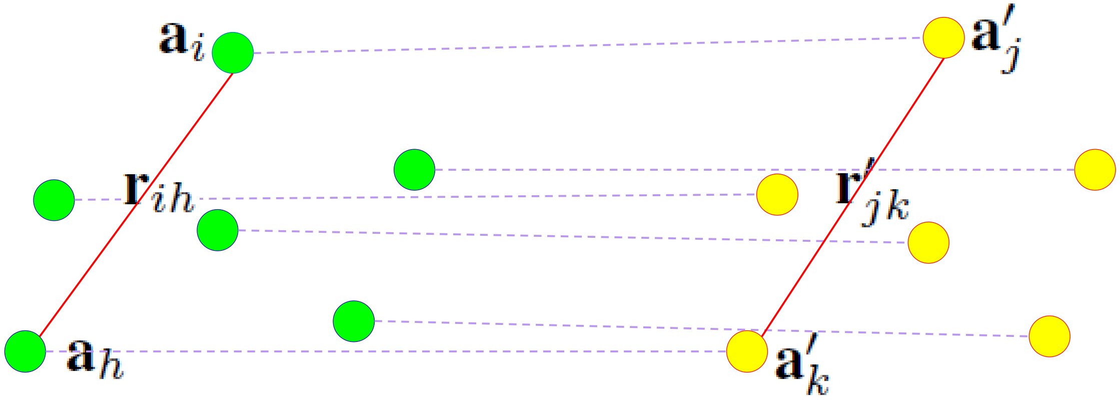
Problem Formulation



Problem Formulation



Problem Formulation



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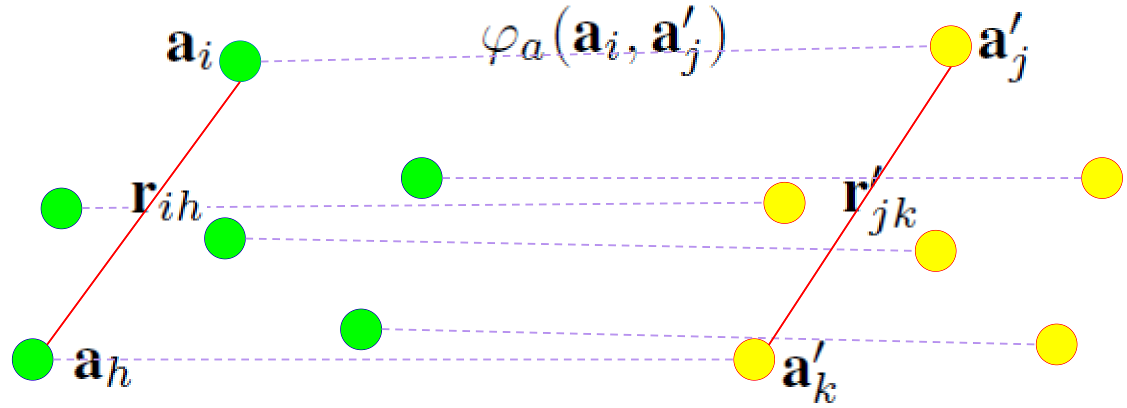

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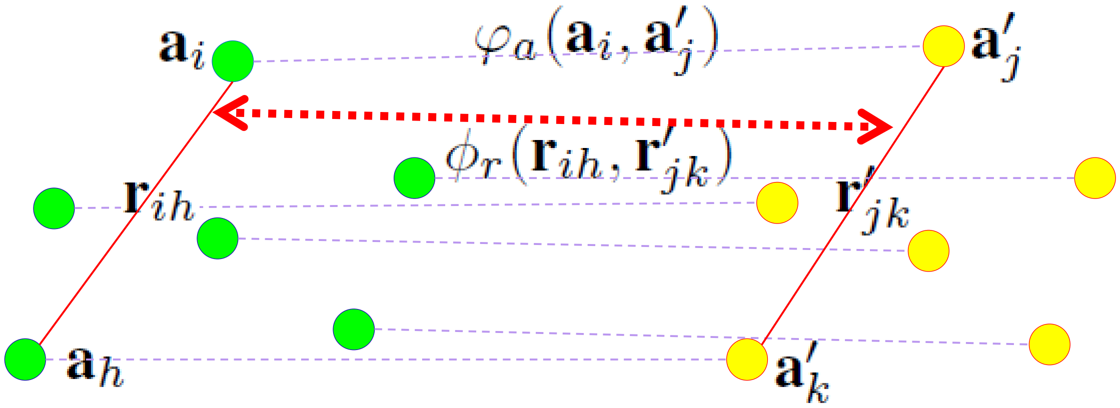

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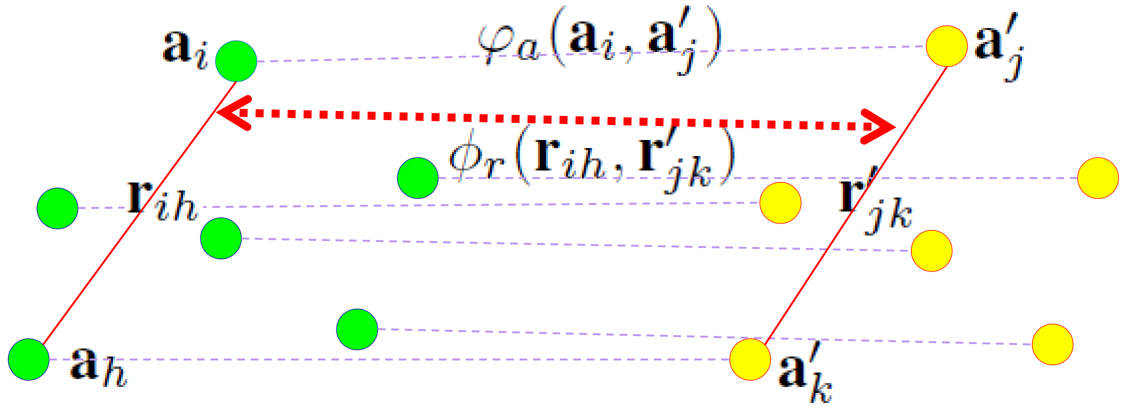
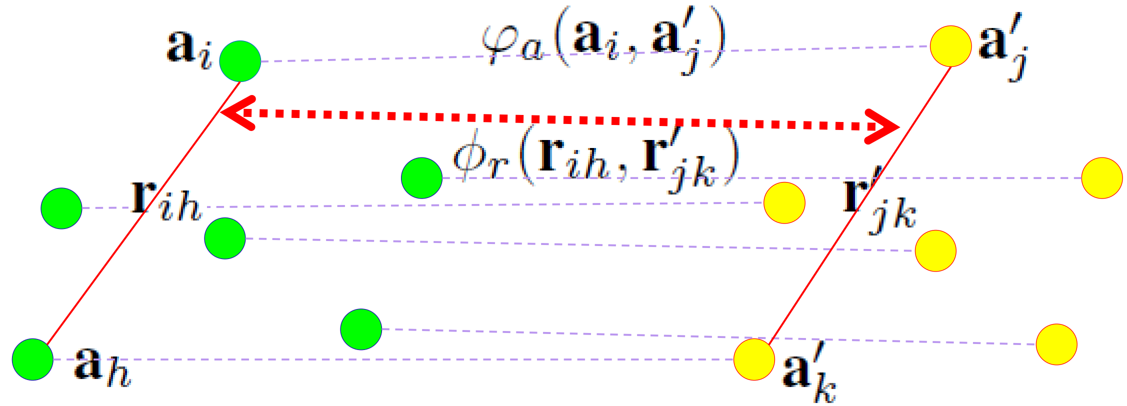


$$\mathbf{X}_{ij} \in \{0, 1\}$$

$$\mathbf{W}_{ij,ij} = \varphi_a(\mathbf{a}_i, \mathbf{a}'_j)$$

$$\mathbf{W}_{ij,hk} = \phi_r(\mathbf{r}_{ih}, \mathbf{r}'_{jk})$$

Problem Formulation



$$\mathbf{X}_{ij} \in \{0, 1\}$$

$$\mathbf{W}_{ij,ij} = \varphi_a(\mathbf{a}_i, \mathbf{a}'_j)$$

$$\mathbf{W}_{ij,hk} = \phi_r(\mathbf{r}_{ih}, \mathbf{r}'_{jk})$$

Integer Quadratic Programming (IQP) problem

$$\max_{\mathbf{X}} \quad \sum\nolimits_{ij} \sum\nolimits_{hk} \mathbf{W}_{ij,hk}\mathbf{X}_{ij}\mathbf{X}_{hk} = \mathrm{vec}(\mathbf{X})^{\mathrm{T}}\mathbf{W}\,\mathrm{vec}(\mathbf{X})$$

$$s.t. \quad \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} \leq \mathbf{1}, \mathbf{X}_{ij} \in \{0, 1\}$$

- NP-hard problem
- Approximate solution

Related works

Continuous Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}(\mathbf{X}) \quad s.t. \ \ \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} \leq \mathbf{1}, \mathbf{X}_{ij} \in \{0,1\}$$

Related works

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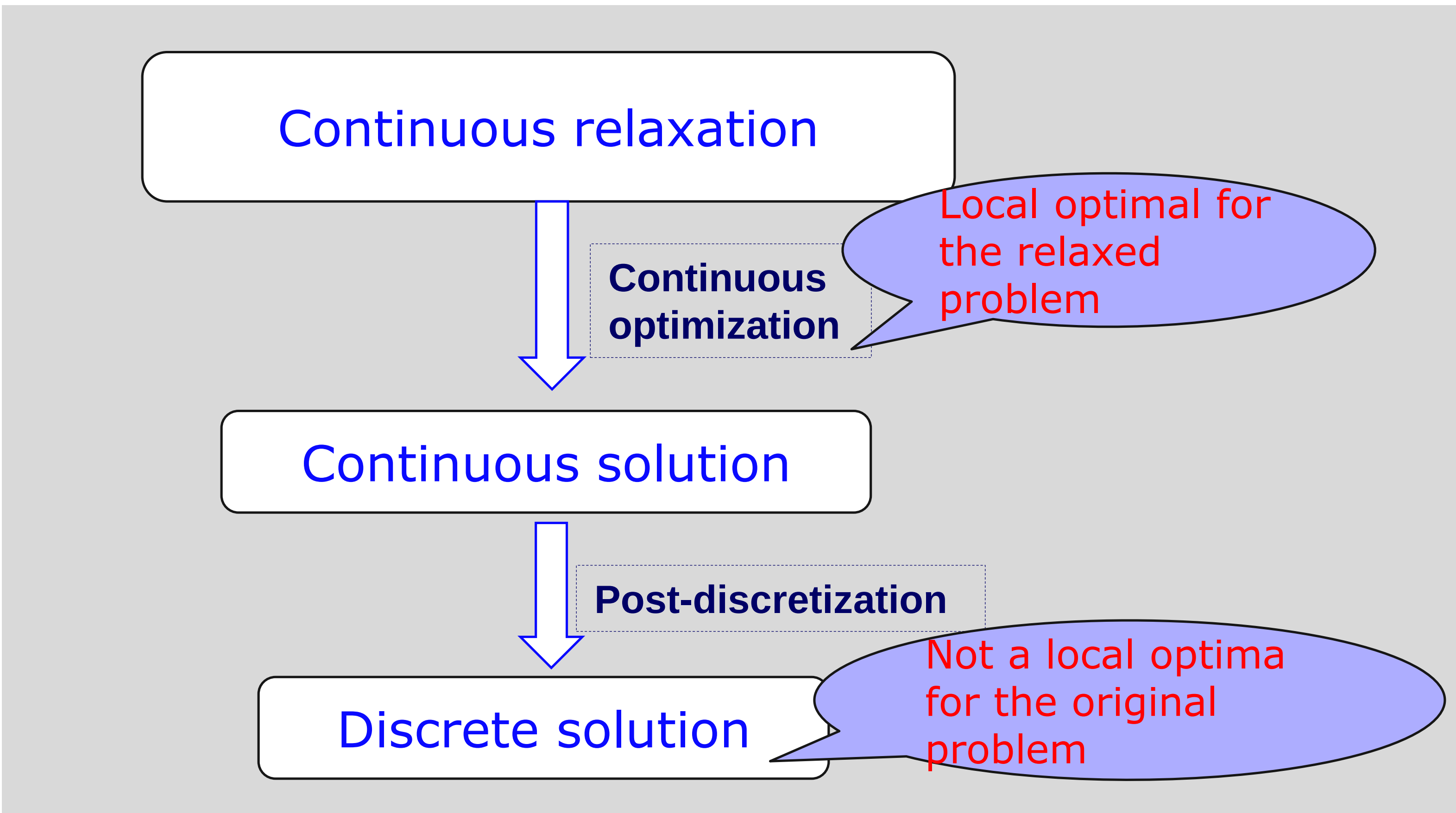
Related works

Continuous Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}}\mathbf{W}\,\mathrm{vec}(\mathbf{X}) \quad s.t. \ \ \mathbf{X}\mathbf{1}=\mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1}\le\mathbf{1}, \mathbf{X}_{ij}\in\{0,1\}$$



Related works

Continuous Relaxation

Spectral matching-ICCV 2005

$$\max \ \operatorname{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \operatorname{vec}(\mathbf{X}) \quad \text{s.t.} \quad \|\mathbf{X}\|_2 = 1$$

Spectral matching with affine constraint-NIPS 2006

$$\max_{\mathbf{X}} \ \operatorname{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{M} \operatorname{vec}(\mathbf{X}) \quad s.t. \ \mathbf{A} \operatorname{vec}(\mathbf{X}) = \mathbf{1}, \ \|\mathbf{X}\|_2^2 = 1$$

Doubly stochastic relaxation

$$\max_{\mathbf{X}} \ \operatorname{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \operatorname{vec}(\mathbf{X})$$

$$s.t. \ \ \mathbf{X}\mathbf{1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} \leq \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

- GA-PAMI 1996
- POCS-PAMI 2004
- RRWM-ICCV 2010
- SCGA-ECCV 2012
- Probabilistic Models

……

Related works

Discrete Methods

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}(\mathbf{X}) \quad s.t. \quad \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} \leq \mathbf{1}, \mathbf{X}_{ij} \in \{0, 1\}$$

Integer Projected Fixed Point (IPFP) -NIPS 2009

Factorized Graph Matching (FGM) - CVPR 2012

$$\max \ J_{\alpha}(\mathbf{X}) = (1-\alpha) J_{vex}(\mathbf{X}) + \alpha J_{cav}(\mathbf{X})$$

Discrete Tabu Search –ICCV 2015

Hungarian-BP-CVPR 2016

Related works

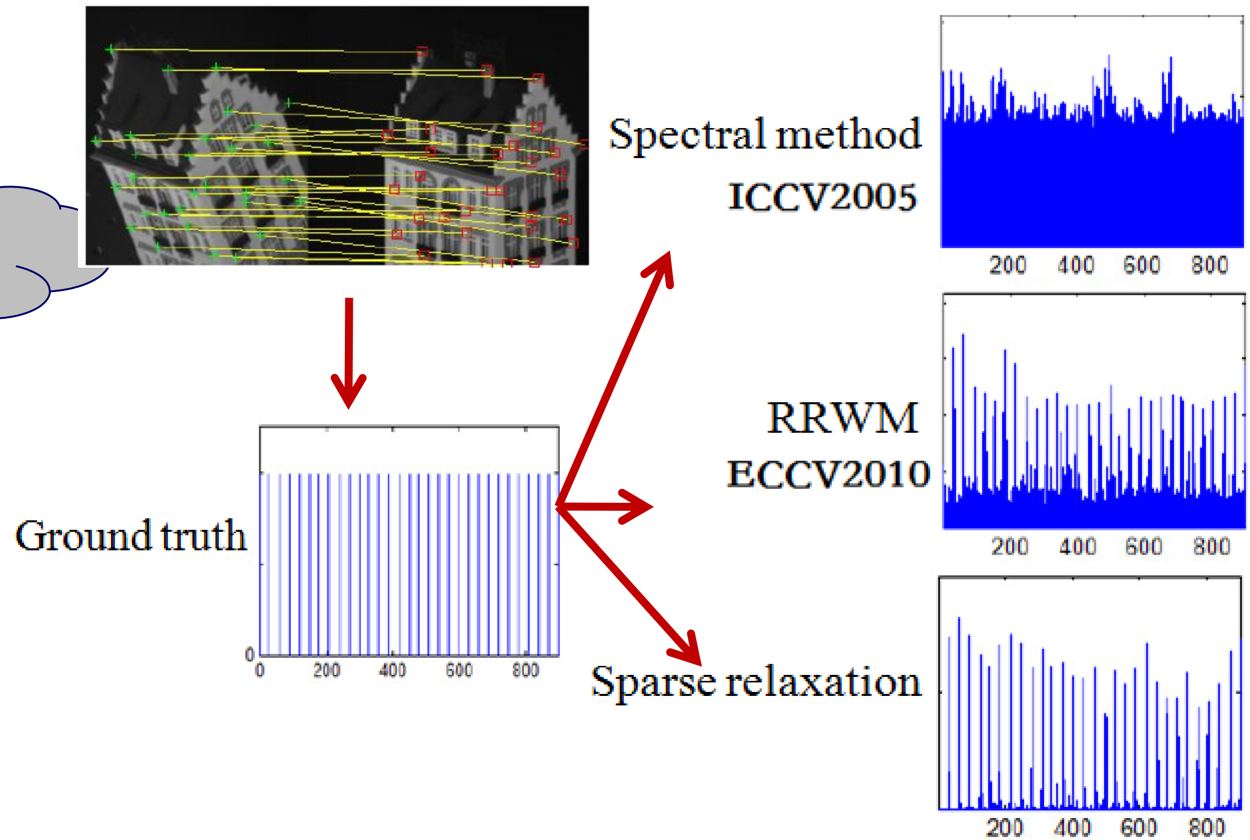
Sparse Relaxation

Discrete constraint

relaxation

Nonnegative sparse

Nonnegative sparse model



Related works

Sparse Relaxation

Spectral matching (SM)-ICCV 2005

$$\max \quad \text{vec}(\mathbf{X})^T \mathbf{W} \, \text{vec}(\mathbf{X}) \quad \text{s.t.} \quad \|\mathbf{X}\|_2 = 1$$

Game-theoretic matching (GameM)-ICCV 2009, IJCV 2011

$$\max \quad \text{vec}(\mathbf{X})^T \mathbf{W} \, \text{vec}(\mathbf{X}) \quad \text{s.t.} \quad \|\mathbf{X}\|_1 = 1, \mathbf{X} \geq 0$$

Elastic net matching (EnetM)-ICCV 2013

$$\max \quad \text{vec}(\mathbf{X})^T \mathbf{W} \, \text{vec}(\mathbf{X})$$

$$\text{s.t.} \quad (1-\alpha)\|\mathbf{X}\|_1 + \alpha\|\mathbf{X}\|_2^2 = 1, \mathbf{X} \geq 0$$

Sparse nonnegative matrix factorization (SNMF)-PR 2014

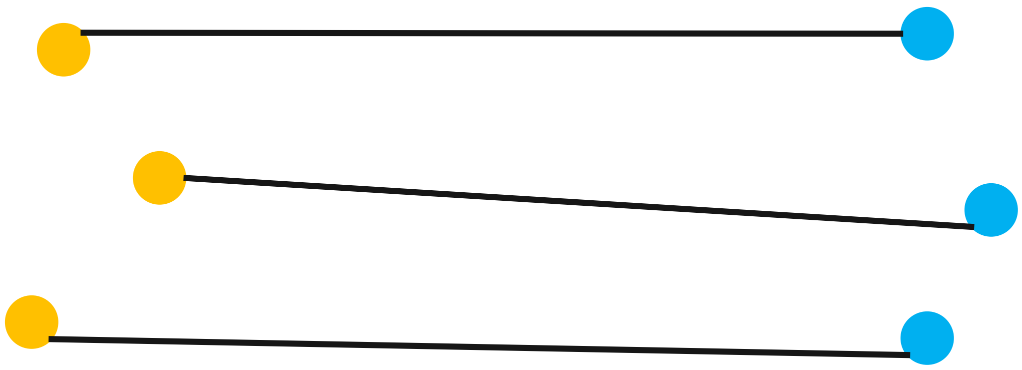
$$\max \quad \text{vec}(\mathbf{X})^T \mathbf{W} \, \text{vec}(\mathbf{X}) \quad \text{s.t.} \quad \|\mathbf{X}\|_p = 1, \mathbf{X} \geq 0$$

Local sparse model for matching

Motivation

Local sparse model for matching

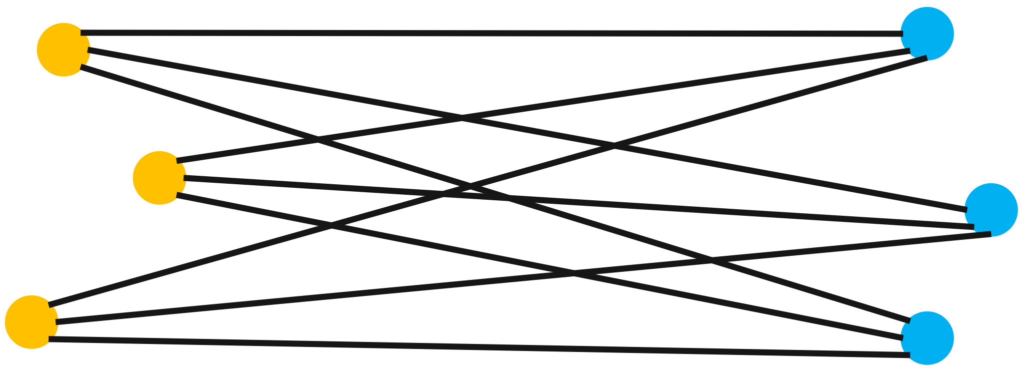
Motivation



$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Local sparse model for matching

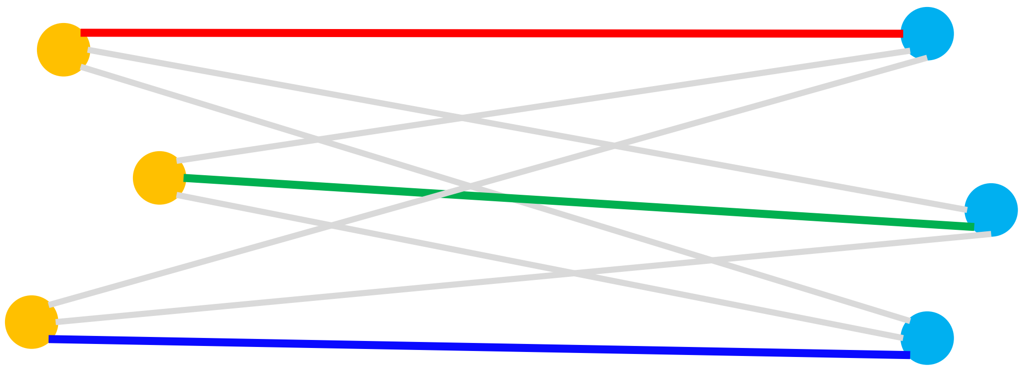
Motivation



$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Local sparse model for matching

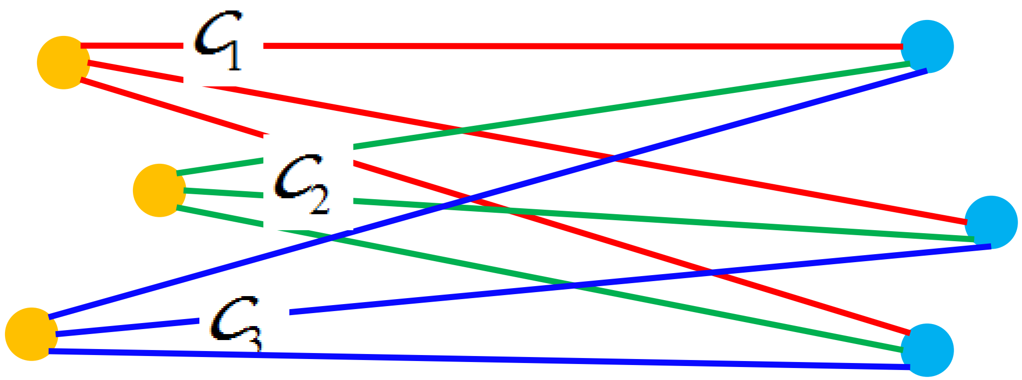
Motivation



$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Local sparse model for matching

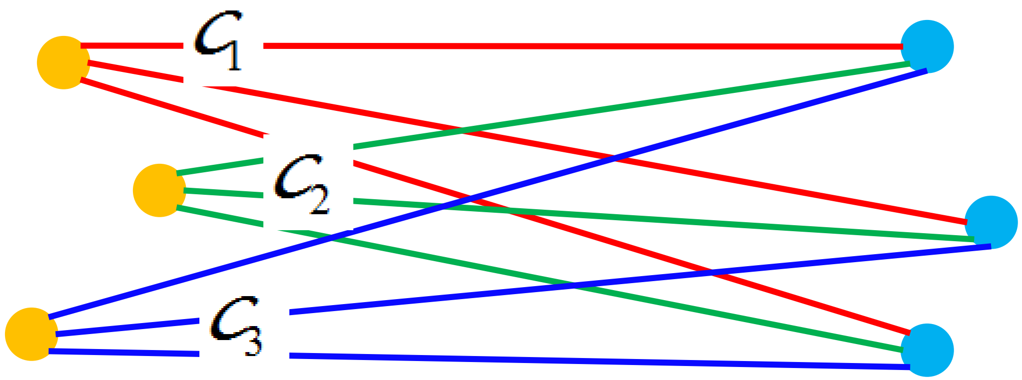
Motivation



$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} C_1 \\ C_2 \\ C_3 \end{matrix}$$

Local sparse model for matching

Motivation



$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} C_1 \\ C_2 \\ C_3 \end{matrix}$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \mathbf{X}_{13} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \mathbf{X}_{23} \\ \mathbf{X}_{31} & \mathbf{X}_{32} & \mathbf{X}_{33} \end{pmatrix}$$

➔ **Local Sparse Model**

Observations

- Each row of solution matrix **X** is sparse
- There is no zero row in solution matrix **X**

Local sparse model for matching

Local sparse matching

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{M} \,\mathrm{vec}(\mathbf{X}) \quad s.t. \ \ \|\mathbf{X}\|_{1,2} = 1, \mathbf{X}_{ij} \geq 0$$

L12 norm

$$\|\mathbf{X}\|_{1,2} = \left[\sum_i \left(\sum_j |\mathbf{X}_{ij}| \right)^2 \right]^{1/2}$$

Local sparse

- L1 norm on each row encourages sparsity
- L2 norm on rows encourages that there is no zero row

Bo Jiang, et al., A Local sparse model for matching problem, **AAAI 2015**

Local sparse model for matching

Algorithm

$$\mathbf{X}_{ij}^{(t+1)} = \mathbf{X}_{ij}^{(t)} \left(\frac{\mathbf{K}_{ij}^{(t)}}{\lambda \sum_i \mathbf{X}_{ij}^{(t)}} \right), \lambda = \mathrm{vec}(\mathbf{X}^{(t)})^T \mathbf{M}\,\mathrm{vec}(\mathbf{X}^{(t)})$$

$\mathbf{K}^{(t)} \in \mathbb{R}^{m\times n}$ is the matrix form of $[\mathbf{W}^{(t)}\,\mathrm{vec}\,(\mathbf{X}^{(t)})]$

Properties

Theorem 1 *Update rule satisfies the first-order Karush-Kuhn-Tucker (KKT) optimality condition.*

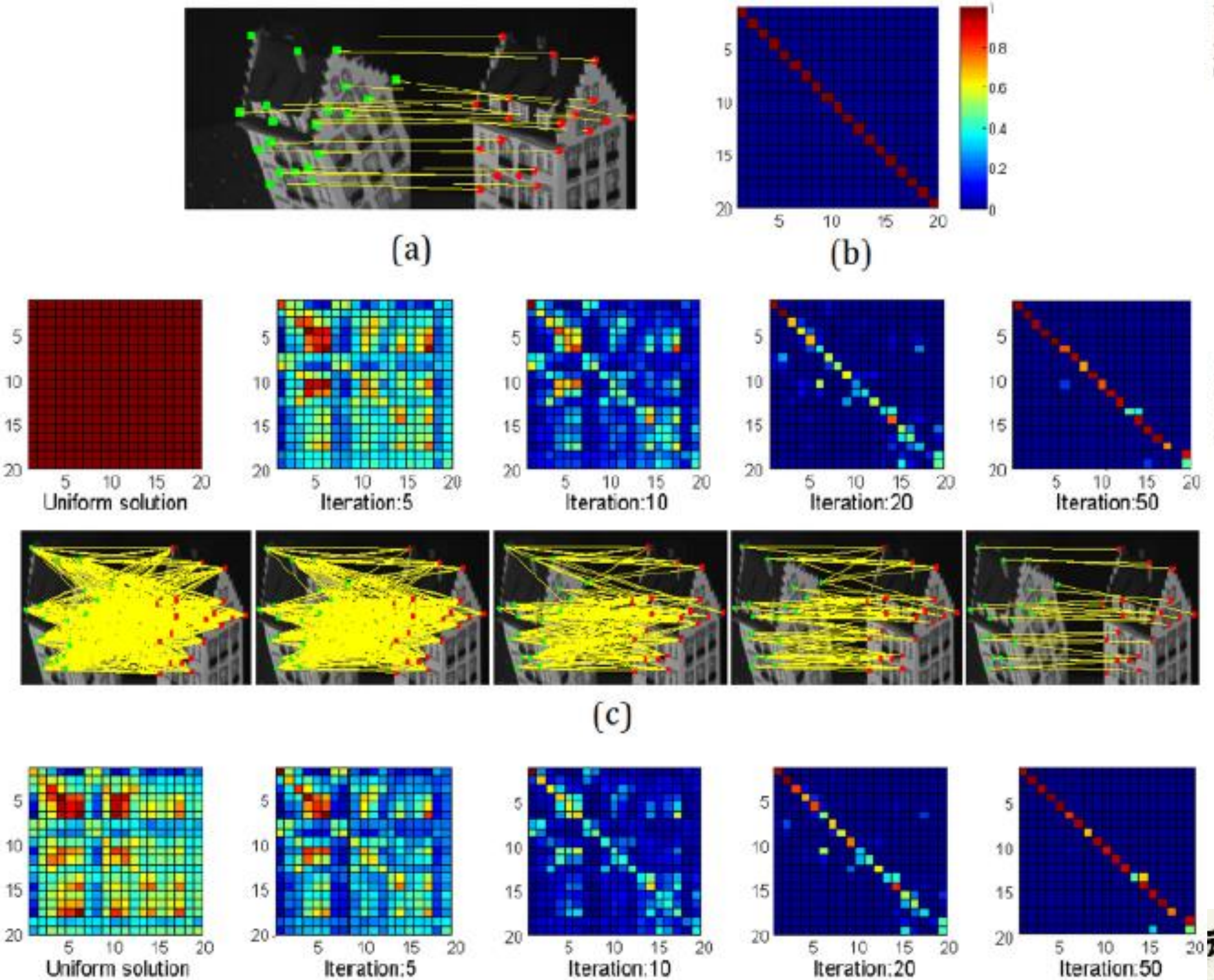
Theorem 2 *Under the update rule, the Lagrangian function $\mathcal{L}(X)$ is monotonically increasing.*

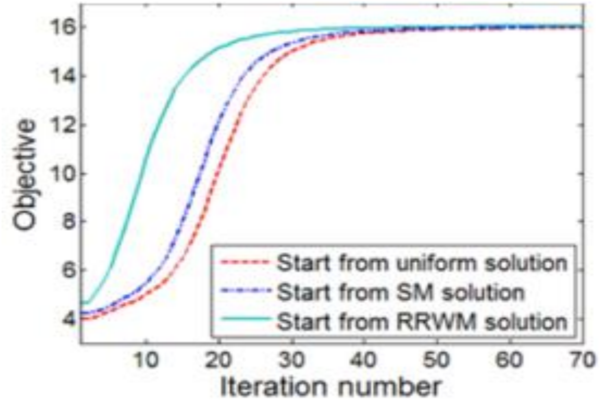
$$\mathcal{L}(\mathbf{X}) = \sum_{ij} \sum_{kl} \mathbf{W}_{ij,kl} \mathbf{X}_{ij} \mathbf{X}_{kl} - \lambda(\mathbf{e}^{\mathrm{T}} \mathbf{X}^{\mathrm{T}} \mathbf{X} \mathbf{e} - 1)$$

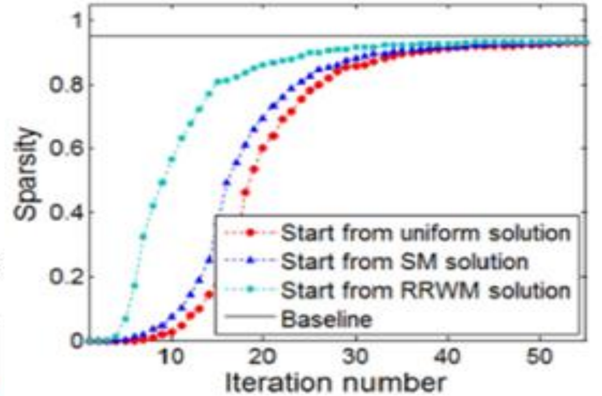
Bo Jiang, et al., A Local sparse model for matching problem, **AAAI 2015**

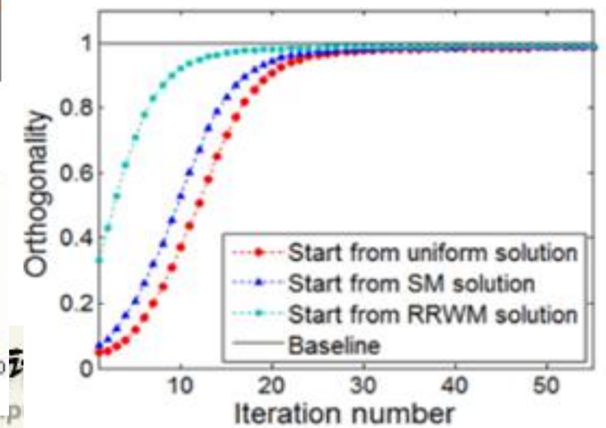
Local sparse model for matching

Illustration

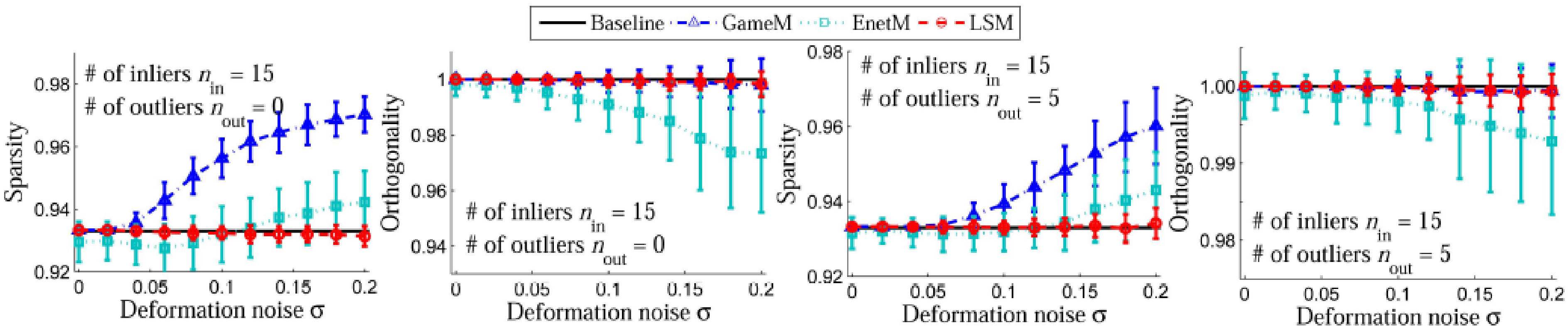


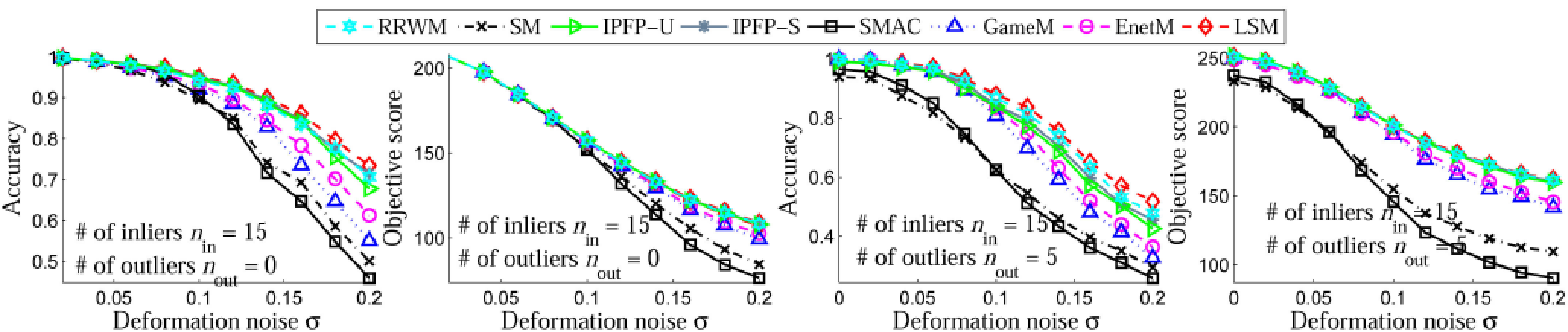






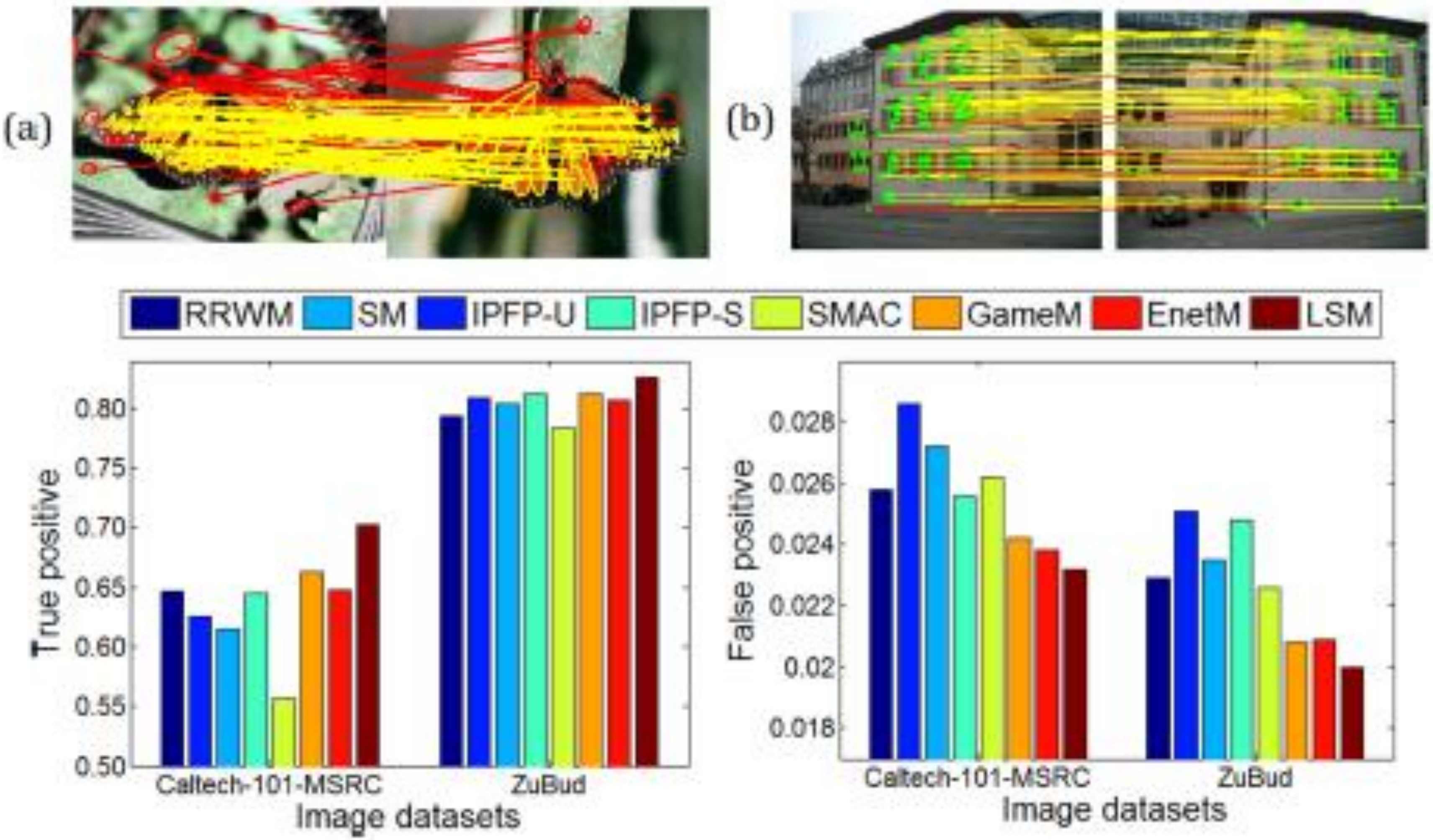
Local sparse model for matching





Bo Jiang, et al., A Local sparse model for matching problem, **AAAI 2015**

Local sparse model for matching



Bo Jiang, et al., A Local sparse model for matching problem, **AAAI 2015**

Binary constraint preserving matching

Motivation

Integer Quadratic Programming (IQP) problem

$$\max_{\mathbf{X}} \quad \sum_{ij} \sum_{hk} \mathbf{W}_{ij,hk} \mathbf{X}_{ij} \mathbf{X}_{hk} = \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}(\mathbf{X})$$

$$s.t. \quad \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} \leq \mathbf{1}, \mathbf{X}_{ij} \in \{0, 1\}$$

$$\max_{\mathbf{X}} \quad \sum_{ij} \sum_{hk} \mathbf{W}_{ij,hk} \mathbf{X}_{ij} \mathbf{X}_{hk} = \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}(\mathbf{X})$$

$$s.t. \quad \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} \leq \mathbf{1}, \|\mathbf{X}\|_2^2 = n$$

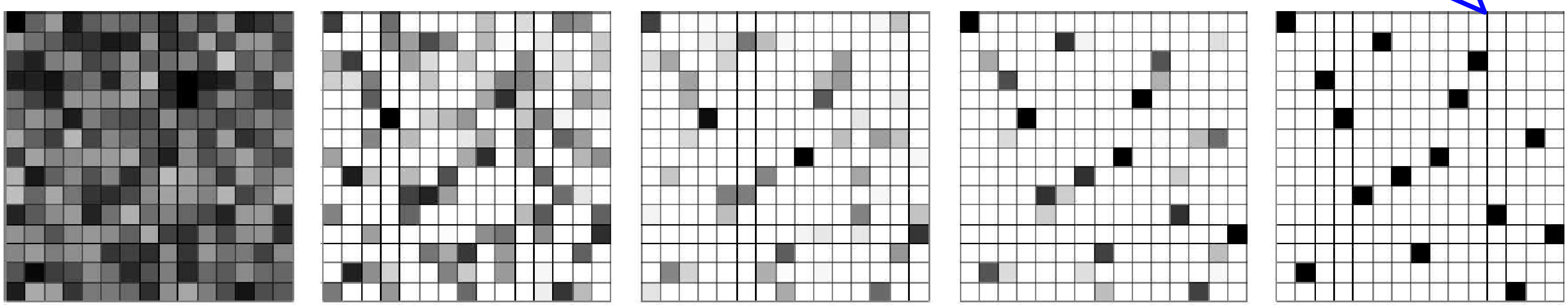
Binary constraint preserving matching

BPGM formulation

$$\max_{\mathbf{x}} \ \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} \quad s.t. \ \mathbf{A}\mathbf{x} = \mathbf{1}, \mathbf{x}_i \geq 0, \|\mathbf{x}\|_2^2 = \gamma.$$

$$\|\mathbf{h}_0\|_2^2 \leq \gamma \leq n, \text{ where } \mathbf{h}_0 = \mathbf{A}^{\mathrm{T}}(\mathbf{A}\,\mathbf{A}^{\mathrm{T}})^{-1}\,\mathbf{1}$$

Binary constraint preserving



$\gamma = 1$ $\gamma = 3$ $\gamma = 7$ $\gamma = 11$ $\gamma = 15$

- As ϒ becomes larger, the more closely **x** approximates to discrete
- It provides a series of relaxation models

Bo Jiang, et al., Binary constraint preserving graph matching, **CVPR 2017**

Binary constraint preserving matching

Theoretical analysis

$$\max_{\mathbf{x}} \ \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} \quad s.t. \ \mathbf{A}\mathbf{x} = \mathbf{1}, \mathbf{x}_i \geq 0, \|\mathbf{x}\|_2^2 = \gamma;$$

Property 1. When $\Upsilon = n$, where n is the number of features, BPGM model is equivalent to original matching problem

$$\max_{\mathbf{x}} \ \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} \quad s.t. \ \mathbf{A}\mathbf{x} = \mathbf{1}, \ \mathbf{x}_i \in \{0, 1\} \qquad (1)$$

Property 2. When $\Upsilon = ||\mathbf{x}^*||$, where $\mathbf{x}^*$ is the optimal solution of problem (2), BPGM model is equivalent to the matching problem (2)

$$\max_{\mathbf{x}} \ \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} \quad s.t. \ \mathbf{A}\mathbf{x} = \mathbf{1}, \ \mathbf{x}_i \geq 0. \qquad (2)$$

Balanced model between (1) and (2)

Bo Jiang, et al., Binary constraint preserving graph matching, **CVPR 2017**

CVPR AHU

Binary constraint preserving matching

Theoretical analysis

$$\max_{\mathbf{x}} \ \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} \quad s.t. \ \mathbf{A}\mathbf{x} = \mathbf{1}, \mathbf{x}_i \geq 0, \|\mathbf{x}\|_2^2 = \gamma.$$

Lemma 3. There exists a parameter Υ_0 such that BPGM with $\Upsilon = \Upsilon_0$ has a global optimal solution

Path-following strategy

$$\max_{\mathbf{x}} \ \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} \quad s.t. \ \mathbf{A}\mathbf{x} = \mathbf{1}, \mathbf{x}_i \geq 0, \|\mathbf{x}\|_2^2 = \gamma_k$$

Starting from global optimal solution and aims to obtain the discrete solution

Bo Jiang, et al., Binary constraint preserving graph matching, **CVPR 2017**

Binary constraint preserving matching

Algorithm

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{v}} \|\mathbf{v} - \widetilde{\mathbf{W}}\mathbf{x}^k\|_2^2$$

$$s.t.\ \ \mathbf{A}\mathbf{v} = \mathbf{1},\ \|\mathbf{v}\|_2^2 = \gamma, \mathbf{v}_i \geq 0$$

$$\widetilde{\mathbf{W}} = \mathbf{W} + \widetilde{\lambda}_m \mathbf{I}$$

$$\mathbf{u} = \widetilde{\mathbf{W}}\mathbf{x}^k$$

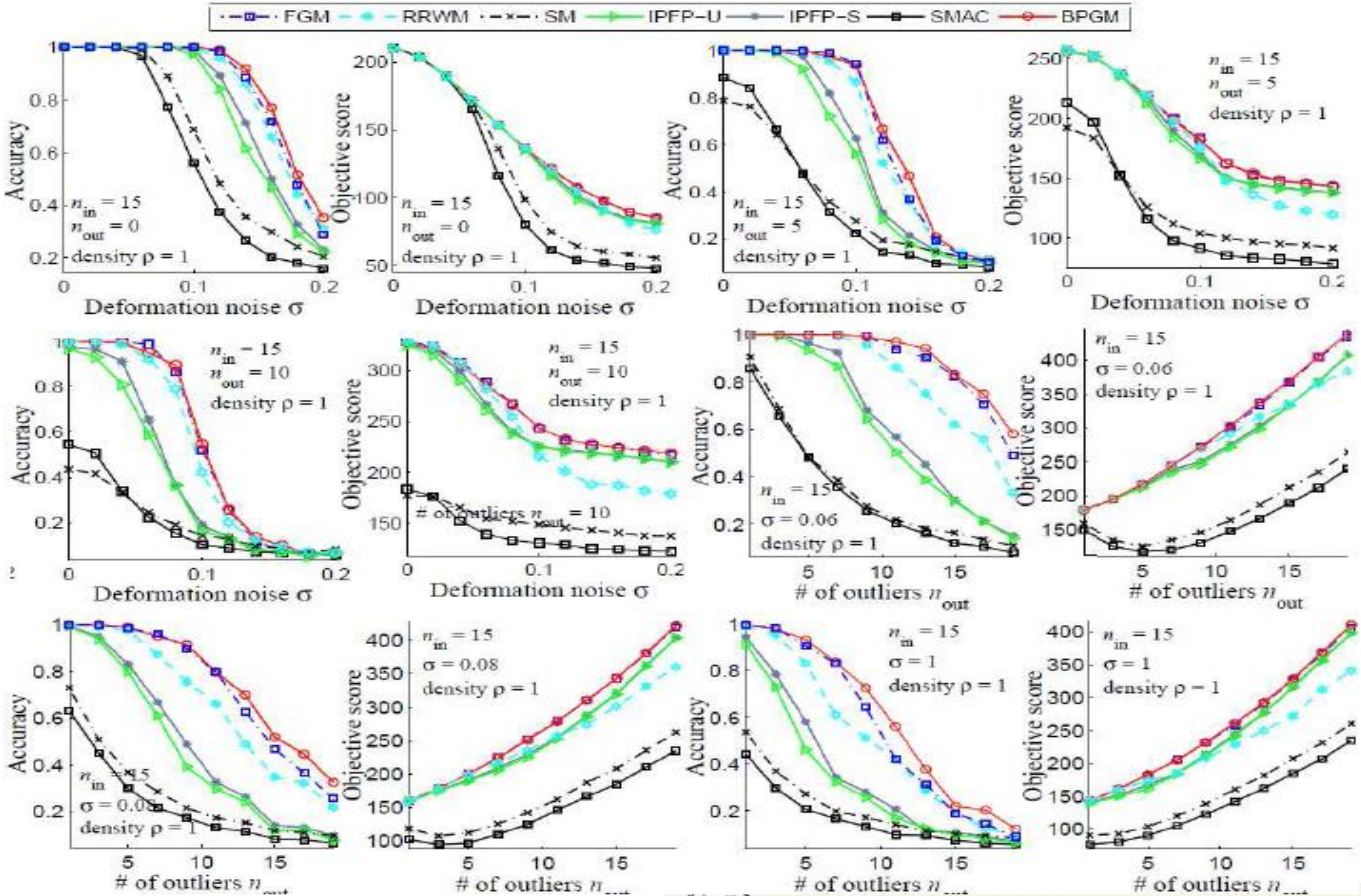
$$P_1(\mathbf{v}) = \arg\min_{\mathbf{v}} \|\mathbf{v} - \mathbf{u}\|_2^2 \quad s.t.\ \ \mathbf{A}\mathbf{v} = \mathbf{1},\ \|\mathbf{v}\|_2^2 = \gamma$$

$$P_2(\mathbf{v}) = \arg\min_{\mathbf{v}} \|\mathbf{v} - \mathbf{u}\|_2^2 \quad s.t.\ \mathbf{v}_i \geq 0$$

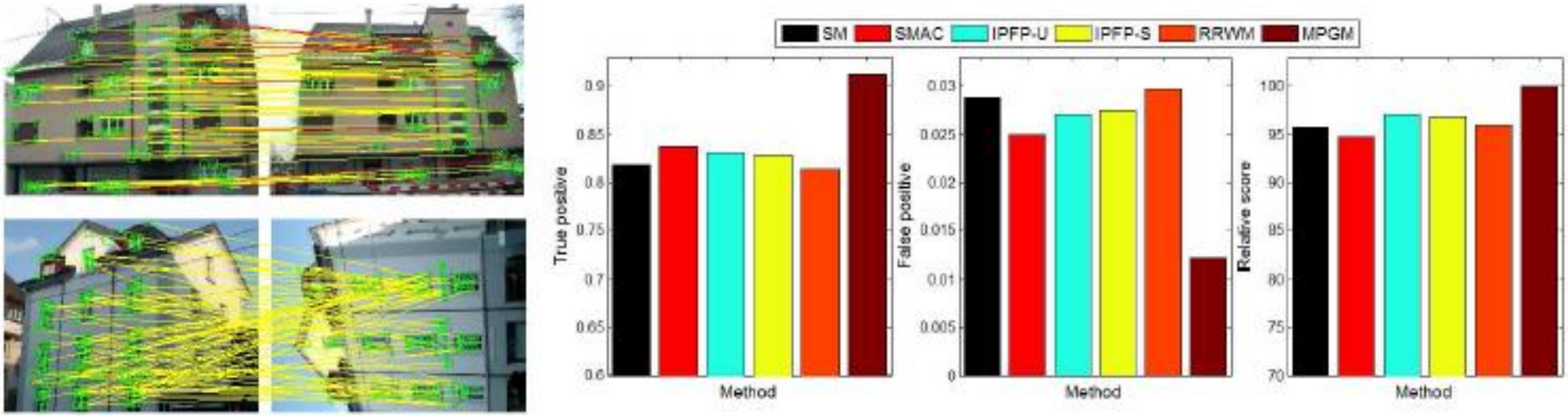
Theorem 1 *Under the update projection Eq.(7), the graph matching objective function* $J_{gm} = \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x}$ *is monotonically increasing.*

Bo Jiang, et al., Binary constraint preserving graph matching, **CVPR 2017**

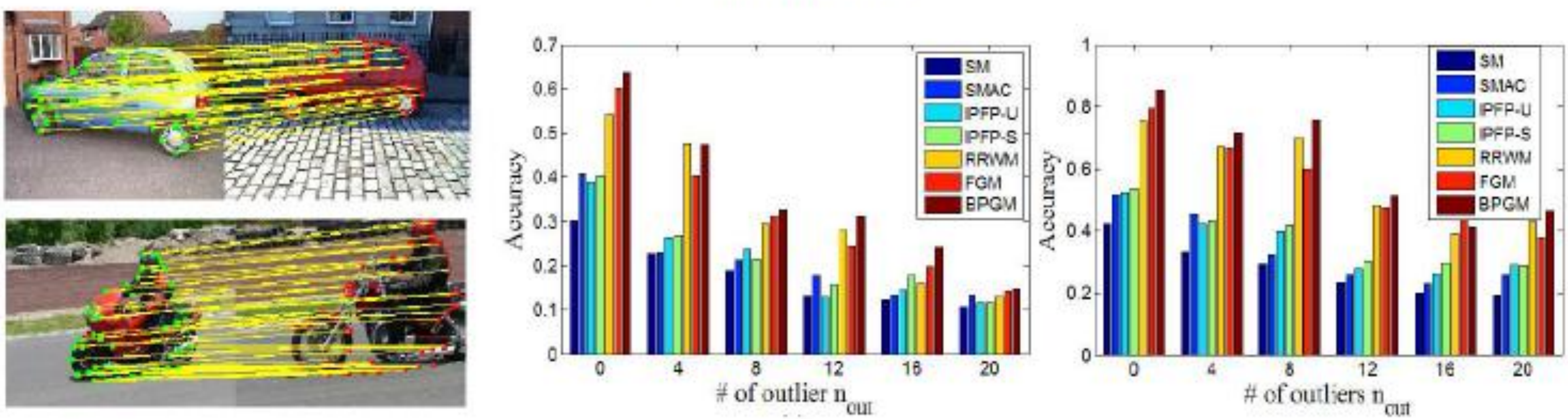
Binary constraint preserving matching



Binary constraint preserving matching



(a) ZuBud image dataset

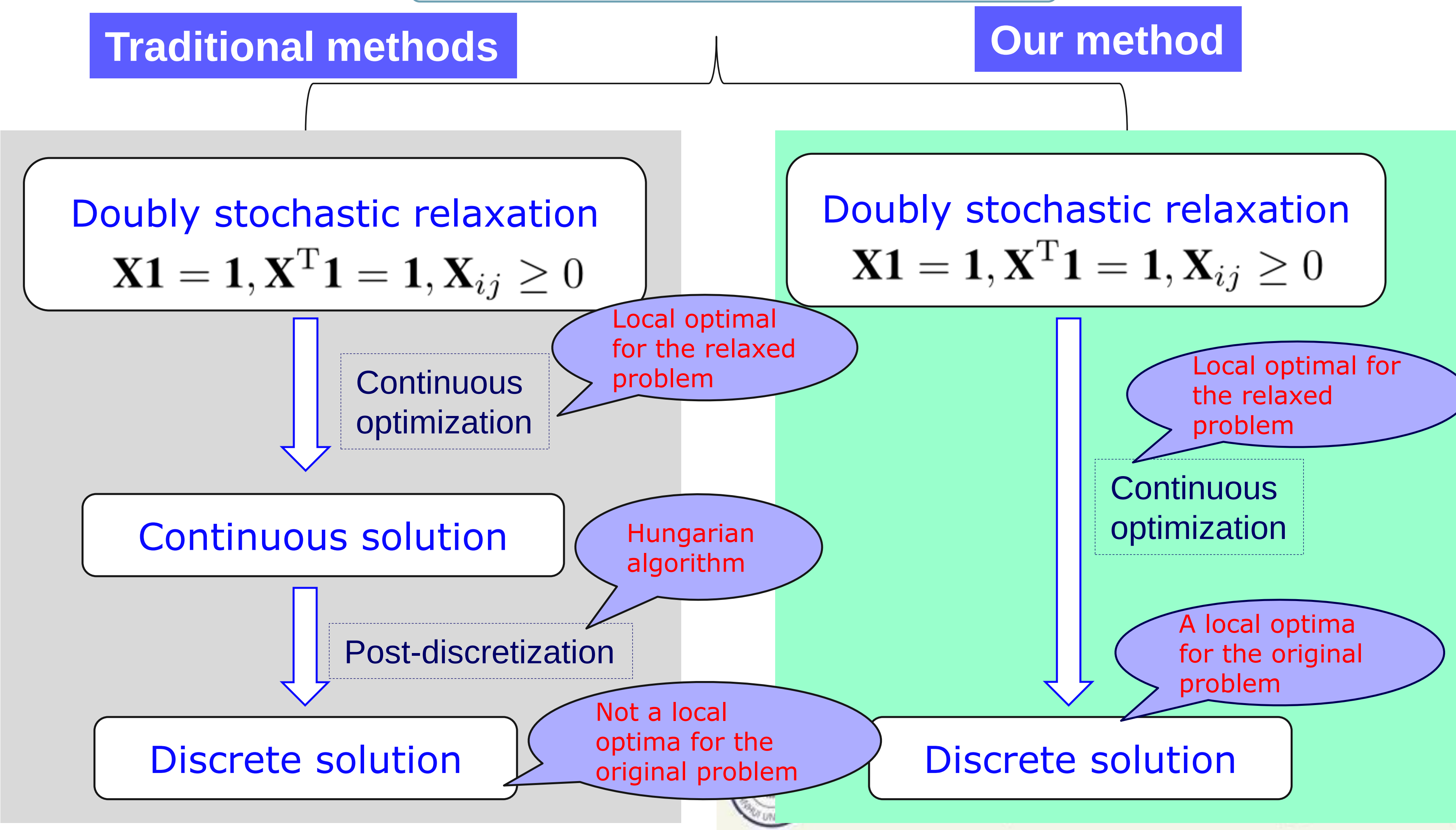


(b) Pascal dataset

Figure 5. Comparison results on real-world image datasets.

Multiplicative update matching

$$\max_{\mathbf{X}} \quad \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W}\, \mathrm{vec}(\mathbf{X})$$
$$s.t. \quad \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1},\ \mathbf{X}_{ij} \in \{0, 1\}$$



Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}(\mathbf{X})$$

$$s.t. \, \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}}\mathbf{W}\,\mathrm{vec}(\mathbf{X})$$

$$s.t.\ \mathbf{X}\mathbf{1}=\mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1}=\mathbf{1}, \mathbf{X}_{ij}\geq 0$$

$$\mathbf{x} = \mathrm{vec}(\mathbf{X})$$

$$\mathcal{L}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} - \sum_{i=1}^{n}\mathbf{\Lambda}_i(\sum_{j=1}^{n}\mathbf{x}_{ij}-1) - \sum_{j=1}^{n}\mathbf{\Gamma}_j(\sum_{i=1}^{n}\mathbf{x}_{ij}-1) - \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{\Delta}_{ij}\mathbf{x}_{ij}$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W}\, \mathrm{vec}(\mathbf{X})$$
$$s.t. \ \mathbf{X}\mathbf{1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

$$\mathbf{x} = \mathrm{vec}(\mathbf{X})$$

$$\mathcal{L}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} \mathbf{W} \mathbf{x} - \sum_{i=1}^{n} \boldsymbol{\Lambda}_i (\sum_{j=1}^{n} \mathbf{x}_{ij} - 1) - \sum_{j=1}^{n} \boldsymbol{\Gamma}_j (\sum_{i=1}^{n} \mathbf{x}_{ij} - 1) - \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\Delta}_{ij} \mathbf{x}_{ij}$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \operatorname{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \operatorname{vec}(\mathbf{X})$$
$$s.t. \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Multipliers

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{x}_{kl}} = 2(\mathbf{W}\mathbf{x})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l - \mathbf{\Delta}_{kl} = 0$$
$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Lambda}_k} = -(\sum_l \mathbf{x}_{kl} - 1) = 0$$
$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Gamma}_l} = -(\sum_k \mathbf{x}_{kl} - 1) = 0$$
$$\mathbf{\Delta}_{kl}\mathbf{x}_{kl} = 0$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}(\mathbf{X})$$

$$s.t. \, \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Multipliers

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{x}_{kl}} = 2(\mathbf{Wx})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l - \mathbf{\Delta}_{kl} = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Lambda}_k} = -(\sum_l \mathbf{x}_{kl} - 1) = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Gamma}_l} = -(\sum_k \mathbf{x}_{kl} - 1) = 0$$

$$\mathbf{\Delta}_{kl}\mathbf{x}_{kl} = 0$$

$$\left[2(\mathbf{Wx})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l\right]\mathbf{x}_{kl} = 0$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}(\mathbf{X})$$

$$s.t. \ \mathbf{X}\mathbf{1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Multipliers

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{x}_{kl}} = 2(\mathbf{W}\mathbf{x})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l - \mathbf{\Delta}_{kl} = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Lambda}_k} = -(\sum_l \mathbf{x}_{kl} - 1) = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Gamma}_l} = -(\sum_k \mathbf{x}_{kl} - 1) = 0$$

$$\mathbf{\Delta}_{kl}\mathbf{x}_{kl} = 0$$

$$\left[2(\mathbf{W}\mathbf{x})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l\right]\mathbf{x}_{kl} = 0$$

$$2\sum_{l=1}^{n} \mathbf{x}_{kl}(\mathbf{W}\mathbf{x})_{kl} - \sum_{l=1}^{n} \mathbf{\Gamma}_l \mathbf{x}_{kl} - \mathbf{\Lambda}_k = 0$$

$$2\sum_{k=1}^{n} \mathbf{x}_{kl}(\mathbf{W}\mathbf{x})_{kl} - \sum_{k=1}^{n} \mathbf{\Lambda}_k \mathbf{x}_{kl} - \mathbf{\Gamma}_l = 0$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}}\mathbf{W}\,\mathrm{vec}(\mathbf{X})$$
$$s.t. \ \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Multipliers

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{x}_{kl}} = 2(\mathbf{Wx})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l - \mathbf{\Delta}_{kl} = 0$$
$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Lambda}_k} = -(\sum_l \mathbf{x}_{kl} - 1) = 0$$
$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Gamma}_l} = -(\sum_k \mathbf{x}_{kl} - 1) = 0$$
$$\mathbf{\Delta}_{kl}\mathbf{x}_{kl} = 0$$

$$\left[2(\mathbf{Wx})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l\right]\mathbf{x}_{kl} = 0$$

$$2\sum_{l=1}^{n} \mathbf{x}_{kl}(\mathbf{Wx})_{kl} - \sum_{l=1}^{n} \mathbf{\Gamma}_l \mathbf{x}_{kl} - \mathbf{\Lambda}_k = 0$$
$$2\sum_{k=1}^{n} \mathbf{x}_{kl}(\mathbf{Wx})_{kl} - \sum_{k=1}^{n} \mathbf{\Lambda}_k \mathbf{x}_{kl} - \mathbf{\Gamma}_l = 0$$

$$2\,\mathrm{diag}(\mathbf{KX}^{\mathrm{T}}) - \mathbf{\Lambda} - \mathbf{X\Gamma} = \mathbf{0},$$
$$2\,\mathrm{diag}(\mathbf{K}^{\mathrm{T}}\mathbf{X}) - \mathbf{\Gamma} - \mathbf{X}^{\mathrm{T}}\mathbf{\Lambda} = \mathbf{0}$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \ \mathrm{vec}(\mathbf{X})^{\mathrm{T}}\mathbf{W}\,\mathrm{vec}(\mathbf{X})$$
$$s.t. \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Multipliers

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{x}_{kl}} = 2(\mathbf{Wx})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l - \mathbf{\Delta}_{kl} = 0$$
$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Lambda}_k} = -(\sum_l \mathbf{x}_{kl} - 1) = 0$$
$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{\Gamma}_l} = -(\sum_k \mathbf{x}_{kl} - 1) = 0$$
$$\mathbf{\Delta}_{kl}\mathbf{x}_{kl} = 0$$

$$\left[2(\mathbf{Wx})_{kl} - \mathbf{\Lambda}_k - \mathbf{\Gamma}_l\right]\mathbf{x}_{kl} = 0$$

$$2\sum_{l=1}^{n} \mathbf{x}_{kl}(\mathbf{Wx})_{kl} - \sum_{l=1}^{n} \mathbf{\Gamma}_l \mathbf{x}_{kl} - \mathbf{\Lambda}_k = 0$$
$$2\sum_{k=1}^{n} \mathbf{x}_{kl}(\mathbf{Wx})_{kl} - \sum_{k=1}^{n} \mathbf{\Lambda}_k \mathbf{x}_{kl} - \mathbf{\Gamma}_l = 0$$

$$2\,\mathrm{diag}(\mathbf{KX}^{\mathrm{T}}) - \mathbf{\Lambda} - \mathbf{X\Gamma} = \mathbf{0},$$
$$2\,\mathrm{diag}(\mathbf{K}^{\mathrm{T}}\mathbf{X}) - \mathbf{\Gamma} - \mathbf{X}^{\mathrm{T}}\mathbf{\Lambda} = \mathbf{0}$$

$$\mathbf{\Gamma} = 2(\mathbf{I} - \mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}(\mathrm{diag}(\mathbf{K}^{\mathrm{T}}\mathbf{X}) - \mathbf{X}^{\mathrm{T}}\mathrm{diag}(\mathbf{KX}^{\mathrm{T}}))$$
$$\mathbf{\Lambda} = 2\,\mathrm{diag}(\mathbf{KX}^{\mathrm{T}}) - \mathbf{X\Gamma}$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \quad \mathrm{vec}\left(\mathbf{X}\right)^{\mathrm{T}} \mathbf{W} \, \mathrm{vec}\left(\mathbf{X}\right)$$

$$s.t. \, \mathbf{X1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Solution update

$$\mathcal{L}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} - \sum_{i=1}^{n} \mathbf{\Lambda}_i \Big(\sum_{j=1}^{n} \mathbf{x}_{ij} - 1\Big) - \sum_{j=1}^{n} \mathbf{\Gamma}_j \Big(\sum_{i=1}^{n} \mathbf{x}_{ij} - 1\Big)$$

Multiplicative update matching

Doubly-stochastic Relaxation

$$\max_{\mathbf{X}} \ \ \mathrm{vec}\,(\mathbf{X})^{\mathrm{T}}\mathbf{W}\,\mathrm{vec}\,(\mathbf{X})$$

$$s.t.\ \mathbf{X}\mathbf{1} = \mathbf{1}, \mathbf{X}^{\mathrm{T}}\mathbf{1} = \mathbf{1}, \mathbf{X}_{ij} \geq 0$$

Solution update

$$\mathcal{L}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} - \sum_{i=1}^{n}\mathbf{\Lambda}_i\Big(\sum_{j=1}^{n}\mathbf{x}_{ij} - 1\Big) - \sum_{j=1}^{n}\mathbf{\Gamma}_j\Big(\sum_{i=1}^{n}\mathbf{x}_{ij} - 1\Big)$$

$$\Phi(\mathbf{x}, \mathbf{x}) = \mathcal{L}(\mathbf{x}), \Phi(\mathbf{x}, \tilde{\mathbf{x}}) \leq \mathcal{L}(\mathbf{x})$$

$$\mathbf{x}^{(t+1)} = \arg\max_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{x}^{(t)})$$

$$\mathcal{L}(\mathbf{x}^{(t)}) = \Phi(\mathbf{x}^{(t)}, \mathbf{x}^{(t)}) \leq \mathcal{L}(\mathbf{x}^{(t+1)})$$

Multiplicative update matching

$$\Phi(\mathbf{x}, \tilde{\mathbf{x}}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{W}_{ij,kl}\tilde{\mathbf{x}}_{ij}\tilde{\mathbf{x}}_{kl}\Big(1 + \log\frac{\mathbf{x}_{ij}\mathbf{x}_{kl}}{\tilde{\mathbf{x}}_{ij}\tilde{\mathbf{x}}_{kl}}\Big)$$

$$-\sum_{i=1}^{n}\mathbf{\Lambda}_i^{+}\Big[\sum_{j=1}^{n}\frac{1}{2}(\frac{\mathbf{x}_{ij}^2}{\tilde{\mathbf{x}}_{ij}} + \tilde{\mathbf{x}}_{ij}) - 1\Big] + \sum_{i=1}^{n}\mathbf{\Lambda}_i^{-}\Big[\sum_{j=1}^{n}\tilde{\mathbf{x}}_{ij}(1 + \log\frac{\mathbf{x}_{ij}}{\tilde{\mathbf{x}}_{ij}}) - 1\Big]$$

$$-\sum_{j=1}^{n}\mathbf{\Gamma}_j^{+}\Big[\sum_{i=1}^{n}\frac{1}{2}(\frac{\mathbf{x}_{ij}^2}{\tilde{\mathbf{x}}_{ij}} + \tilde{\mathbf{x}}_{ij}) - 1\Big] + \sum_{j=1}^{n}\mathbf{\Gamma}_j^{-}\Big[\sum_{i=1}^{n}\tilde{\mathbf{x}}_{ij}(1 + \log\frac{\mathbf{x}_{ij}}{\tilde{\mathbf{x}}_{ij}}) - 1\Big]$$

Multiplicative update matching

$$\Phi(\mathbf{x},\tilde{\mathbf{x}}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{W}_{ij,kl}\tilde{\mathbf{x}}_{ij}\tilde{\mathbf{x}}_{kl}\Big(1+\log\frac{\mathbf{x}_{ij}\mathbf{x}_{kl}}{\tilde{\mathbf{x}}_{ij}\tilde{\mathbf{x}}_{kl}}\Big)$$

$$-\sum_{i=1}^{n}\mathbf{\Lambda}_i^{+}\Big[\sum_{j=1}^{n}\frac{1}{2}\Big(\frac{\mathbf{x}_{ij}^2}{\tilde{\mathbf{x}}_{ij}}+\tilde{\mathbf{x}}_{ij}\Big)-1\Big]+\sum_{i=1}^{n}\mathbf{\Lambda}_i^{-}\Big[\sum_{j=1}^{n}\tilde{\mathbf{x}}_{ij}\Big(1+\log\frac{\mathbf{x}_{ij}}{\tilde{\mathbf{x}}_{ij}}\Big)-1\Big]$$

$$-\sum_{j=1}^{n}\mathbf{\Gamma}_j^{+}\Big[\sum_{i=1}^{n}\frac{1}{2}\Big(\frac{\mathbf{x}_{ij}^2}{\tilde{\mathbf{x}}_{ij}}+\tilde{\mathbf{x}}_{ij}\Big)-1\Big]+\sum_{j=1}^{n}\mathbf{\Gamma}_j^{-}\Big[\sum_{i=1}^{n}\tilde{\mathbf{x}}_{ij}\Big(1+\log\frac{\mathbf{x}_{ij}}{\tilde{\mathbf{x}}_{ij}}\Big)-1\Big]$$

Algorithm

$$\mathbf{x}^{(t+1)} = \arg\max_{\mathbf{x}} \Phi(\mathbf{x},\mathbf{x}^{(t)})$$

$$\mathbf{x}_{kl} = \tilde{\mathbf{x}}_{kl}\Big[\frac{2(\mathbf{W}\tilde{\mathbf{x}})_{kl}+\mathbf{\Lambda}_k^{-}+\mathbf{\Gamma}_l^{-}}{\mathbf{\Lambda}_k^{+}+\mathbf{\Gamma}_l^{+}}\Big]^{1/2}$$

Multiplicative update matching

$$\begin{aligned}\Phi(\mathbf{x},\tilde{\mathbf{x}}) = & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\mathbf{W}_{ij,kl}\tilde{\mathbf{x}}_{ij}\tilde{\mathbf{x}}_{kl}\Big(1+\log\frac{\mathbf{x}_{ij}\mathbf{x}_{kl}}{\tilde{\mathbf{x}}_{ij}\tilde{\mathbf{x}}_{kl}}\Big)\\ & -\sum_{i=1}^{n}\mathbf{\Lambda}_i^{+}\Big[\sum_{j=1}^{n}\frac{1}{2}\big(\frac{\mathbf{x}_{ij}^2}{\tilde{\mathbf{x}}_{ij}}+\tilde{\mathbf{x}}_{ij}\big)-1\Big]+\sum_{i=1}^{n}\mathbf{\Lambda}_i^{-}\Big[\sum_{j=1}^{n}\tilde{\mathbf{x}}_{ij}\big(1+\log\frac{\mathbf{x}_{ij}}{\tilde{\mathbf{x}}_{ij}}\big)-1\Big]\\ & -\sum_{j=1}^{n}\mathbf{\Gamma}_j^{+}\Big[\sum_{i=1}^{n}\frac{1}{2}\big(\frac{\mathbf{x}_{ij}^2}{\tilde{\mathbf{x}}_{ij}}+\tilde{\mathbf{x}}_{ij}\big)-1\Big]+\sum_{j=1}^{n}\mathbf{\Gamma}_j^{-}\Big[\sum_{i=1}^{n}\tilde{\mathbf{x}}_{ij}\big(1+\log\frac{\mathbf{x}_{ij}}{\tilde{\mathbf{x}}_{ij}}\big)-1\Big]\end{aligned}$$

Algorithm

$$\mathbf{x}^{(t+1)} = \arg\max_{\mathbf{x}} \Phi(\mathbf{x},\mathbf{x}^{(t)})$$

$$\mathbf{x}_{kl} = \tilde{\mathbf{x}}_{kl}\Big[\frac{2(\mathbf{W}\tilde{\mathbf{x}})_{kl}+\mathbf{\Lambda}_k^{-}+\mathbf{\Gamma}_l^{-}}{\mathbf{\Lambda}_k^{+}+\mathbf{\Gamma}_l^{+}}\Big]^{1/2}$$

$$\begin{aligned}\mathbf{\Gamma} = & 2\big(\mathbf{I}-{\mathbf{X}^{(t)}}^{\mathrm{T}}\mathbf{X}^{(t)}\big)^{-1}\Big[\mathrm{diag}\,\big({\mathbf{K}^{(t)}}^{\mathrm{T}}\mathbf{X}^{(t)}\big)-{\mathbf{X}^{(t)}}^{\mathrm{T}}\,\mathrm{diag}\,\big(\mathbf{K}^{(t)}{\mathbf{X}^{(t)}}^{\mathrm{T}}\big)\Big]\\ \mathbf{\Lambda} = & 2\,\mathrm{diag}\,\big(\mathbf{K}^{(t)}{\mathbf{X}^{(t)}}^{\mathrm{T}}\big)-\mathbf{X}^{(t)}\mathbf{\Gamma}\end{aligned}$$

Multiplicative update matching

Convergence

Theorem 1. *Under update, the Lagrangian function $\mathcal{L}(\mathbf{x})$ is monotonically increasing,*

$$\mathcal{L}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{W}\mathbf{x} - \sum_{i=1}^{n} \mathbf{\Lambda}_i \Big(\sum_{j=1}^{n} \mathbf{x}_{ij} - 1\Big) - \sum_{j=1}^{n} \mathbf{\Gamma}_j \Big(\sum_{i=1}^{n} \mathbf{x}_{ij} - 1\Big)$$

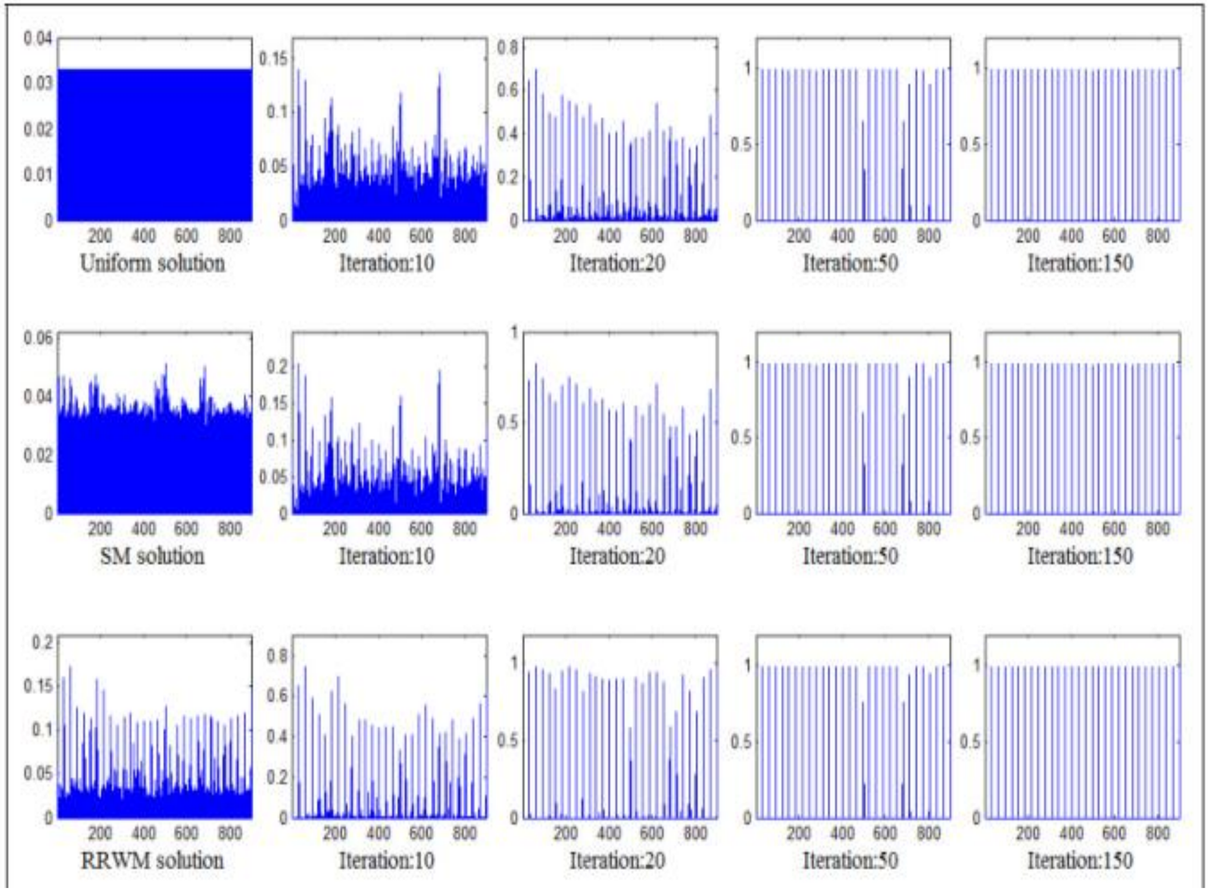
where $\mathbf{\Lambda}, \mathbf{\Gamma}$ are Lagrangian multipliers.

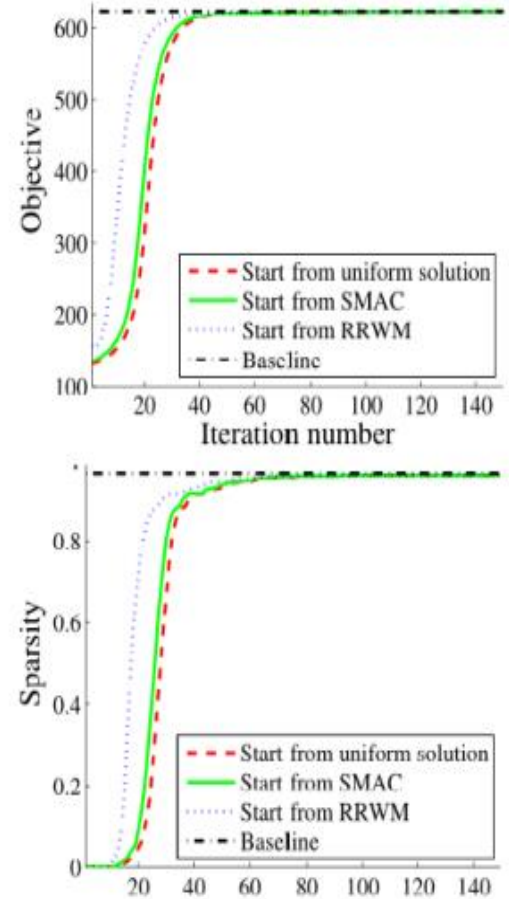
Optimality

Theorem 2. *Under update, the converged solution $\mathbf{x}^*$ is Karush-Kuhn Tucker (KKT) optimal.*

Bo Jiang, et al., Graph Matching via Multiplicative Update Algorithm, **NIPS** 2017

Multiplicative update matching





Top: start from uniform solution
Middle: start from Spectral Matching (SM) solution
Bottom: start from Random Walk (RRWM) solution

Bo Jiang, et al., Graph Matching via Multiplicative Update Algorithm, **NIPS** 2017

Multiplicative update matching

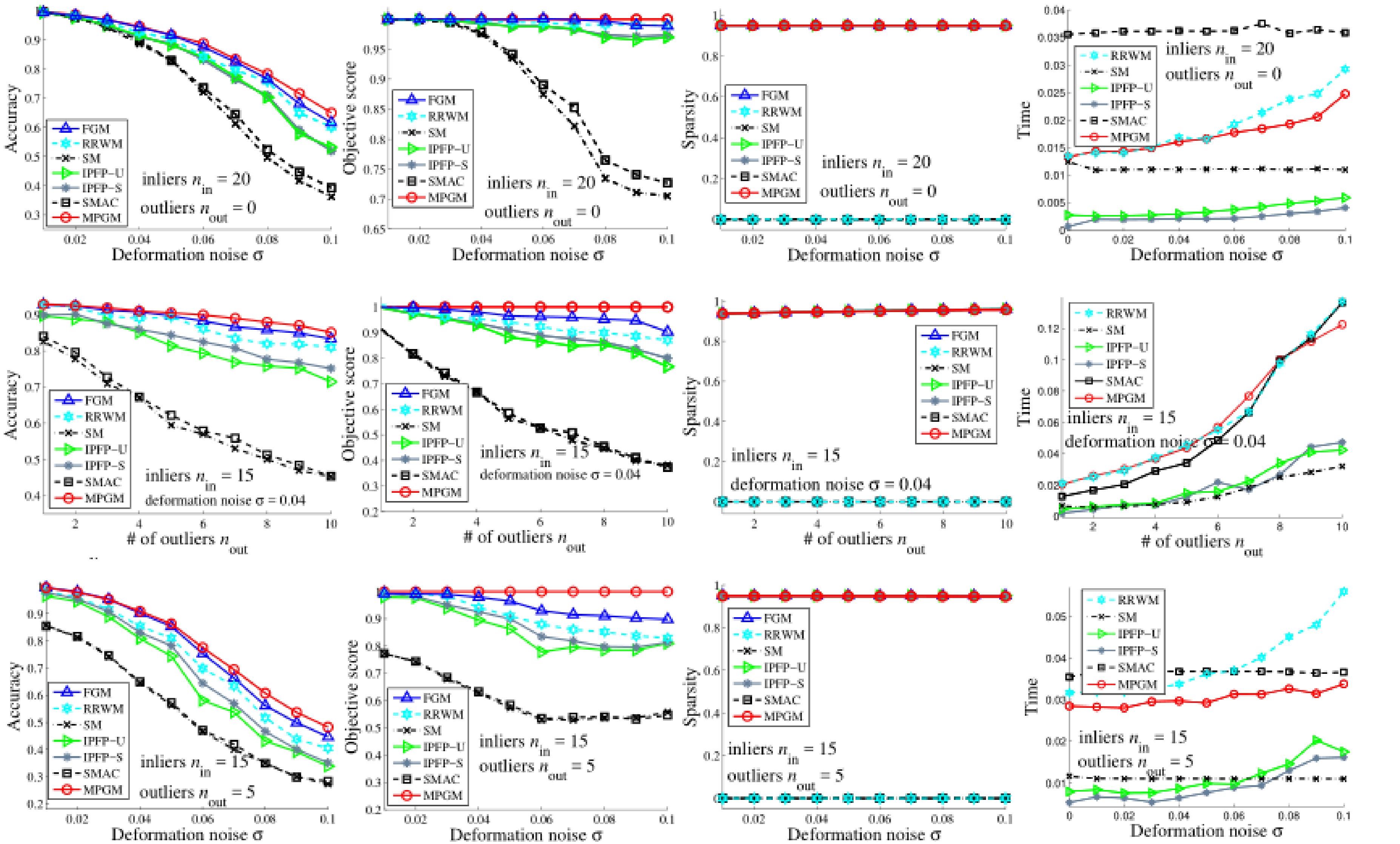


Figure 2: Comparison results of different methods on synthetic point sets matching

Multiplicative update matching

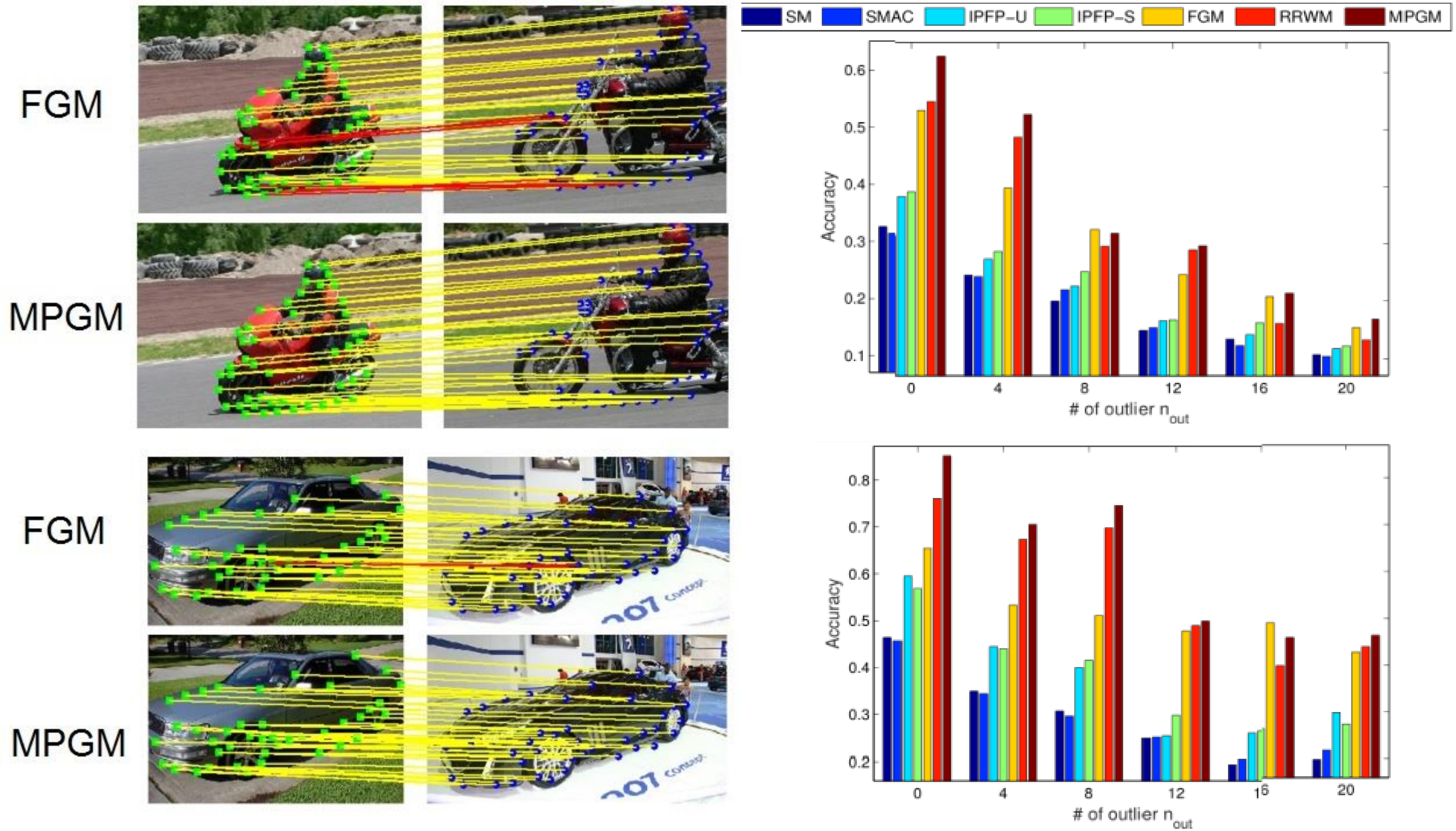


Figure 5: Comparison results of different graph matching methods on the Pascal 2007 dataset

Reference

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- Bo Jiang, Jin Tang, Chris Ding and Bin Luo, A local sparse model for matching problem, AAAI Conference on Artificial Intelligence (*AAAI*), pp. 3790-3796, 2015
- Bo Jiang, Jin Tang, Bin Luo and Liang Lin, Robust feature point matching with sparse model, *IEEE Transactions on Image Processing*, 23(12):5175-5186, 2014

Conclusion and Future works

Conclusion

- Sparse relaxation model for matching problem
- Binary constraint preserving model for matching
- Multiplicative update algorithm for matching

Future works

- More theoretical analysis on Multiplicative matching
- More effective algorithm to solve sparse matching model
- Matching objective relaxation

Thank you !

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